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NOTE ON THE DERIVATIVE OF THE QUOTIENT OF TWO WRONSKIANS.

By L. A. HOWLAND, Middletown, Connecticut.

Suppose we have given a set of $n+1$ analytic functions of x, in which there are two sets of n functions each, which are linearly independent. A formula for the derivative of the quotient of the Wronskians of these two sets has important application in the theory of linear differential equations and elsewhere.* The formula is supposed to be due to Frobenius, who derived and applied it in the article cited. It may be of interest to note that we can reverse the application and use the theory of linear differential equations to establish the formula.

Since it is merely a question of notation, we shall assume the sets to be: (1) the first n functions, and (2) the first $n-1$ and the last. We have then to evaluate

$$(1)\qquad \frac{d}{dx}\,\frac{W_2(a_1,\, a_2,\, \dots,\, a_{n-1},\, a_{n+1})}{W_1(a_1,\, a_2,\, \dots,\, a_{n-1},\, a_n)}=\frac{W_1W_2'-W_1'W_2}{W_1^2}$$

where $W_1=W_1(a_1,\, a_2,\, \dots,\, a_n)=\begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_1' & a_2' & \dots & a_n' \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(n-1)} & a_1^{(n-1)} & \dots & a_n^{(n-1)} \end{vmatrix}$

and similarly for W_2.

Form the equation:

$$(2)\qquad W_1W_2'-W_1'W_2-W(a_1,\, \dots,\, a_{n-1})\,W(a_1,\, \dots,\, a_{n+1})=0,$$

or written out in determinant form:

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* Cf., for example, Frobenius, Crelle, 77, p. 248; Schlesinger, Handbuch d. Theorie d. Lin. Diff. Gl., Bd. 1, p. 60; Curtiss, Math. Ann., Bd. 65, p. 284; also Bull. Am. Math. Soc., Vol. XVII, p. 465.

$$\begin{vmatrix} a_1 & \dots & a_n \\ a_1' & \dots & a_n' \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_1^{(n-1)} & \dots & a_n^{(n-1)} \end{vmatrix} \begin{vmatrix} a_1 & \dots & a_{n-1} & a_{n+1} \\ a_1' & \dots & a'_{n-1} & a'_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(n-2)} & \dots & a_{n-1}^{(n-2)} & a_{n+1}^{(n-2)} \\ a_1^{(n)} & \dots & a_{n-1}^{(n)} & a_{n+1}^{(n)} \end{vmatrix} - \begin{vmatrix} a_1 & \dots & a_n \\ a_1' & \dots & a_n' \\ \cdot & \cdot & \cdot \\ a_1^{(n-2)} & \dots & a_n^{(n-2)} \\ a_1^{(n)} & \dots & a_n^{(n)} \end{vmatrix}$$

$$\begin{vmatrix} a_1 & \dots & a_{n-1} & a_{n+1} \\ a_1' & \dots & a'_{n-1} & a'_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(n-1)} & \dots & a_{n-1}^{(n-1)} & a_{n+1}^{(n-1)} \end{vmatrix} - \begin{vmatrix} a_1 & \dots & a_{n-1} \\ a_1' & \dots & a'_{n-2} \\ \cdot & \cdot & \cdot \\ a_1^{(n-2)} & \dots & a_{n-1}^{(n-2)} \end{vmatrix} \begin{vmatrix} a_1 & \dots & a_{n+1} \\ a_1' & \dots & a'_{n+1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_1^{(n-1)} & \dots & a_{n+1}^{(n-1)} \\ a_1^{(n)} & \dots & a_{n+1}^{(n)} \end{vmatrix} = 0.$$

This is obviously a homogeneous linear differential equation in a_{n+1} of order at most equal to n. Its order, however, is seen by inspection to be less than n, for the coefficient of $a_{n+1}^{(n)}$ in the expansion is

$$W(a_1 \dots a_n)\, W(a_1 \dots a_{n-1}) - W(a_1 \dots a_{n-1})\, W(a_1 \dots a_n) \equiv 0.$$

Again by inspection it is easily seen that the equation has the n integrals a_1, $a_2 \dots a_n$, for the second determinant in each term has two columns alike when a_{n+1} is put equal to a_1, $a_2 \dots a_{n-1}$, whereas when $a_{n+1} = a_n$, the first two terms become identical and the last vanishes as before. But a linear equation of this sort and of order less than n which has n linearly independent integrals vanishes identically and hence (2) is an identity.

We have thus the formula:

$$\frac{d}{dx} \frac{W_2(a_1 \dots a_{n-1}\, a_{n+1})}{W_1(a_1 \dots a_{n-1}\, a_n)} = \frac{W(a_1 \dots a_{n-1})\, W(a_1 \dots a_{n+1})}{[W(a_1 \dots a_n)]^2}.$$

Finally, we will note concerning this proof, as did Professor Curtiss concerning the derivation of the formula by Frobenius,* that although we started with analytic functions, the formula holds for any functions having the necessary derivatives. In fact, formula (2) holds when the primes, etc., lose the significance of differentiation, for the vanishing of the first member of (2) is purely formal; by that we mean that it vanishes identically when regarded as a polynomial in the $(n+1)^2$ independent variables $a_1 \dots a_{n+1}$, $a_1' \dots a'_{n+1} \dots a_{n+1}^{(n)}$, since the first $n+1$ of these, the original functions, may be so chosen that each of these symbols has at a given point any prescribed value. As an example:

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* Math. Ann., Bd. 65, foot-note on p. 284.

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & a_{41} \\ a_{12} & a_{22} & a_{42} \\ a_{14} & a_{24} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{14} & a_{24} & a_{34} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & a_{41} \\ a_{12} & a_{22} & a_{42} \\ a_{13} & a_{23} & a_{43} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}$$

ON A SPECIAL CASE OF EQUILIBRIUM OF A FLEXIBLE, INEXTENSIBLE STRING.

By CHARLES C. STECK, Durham, New Hampshire.

It is proposed in this paper to discuss the curve of equilibrium of a weightless, flexible, inextensible string, the end points of which are attached to two fixed points, each element of the string being acted upon by a force always normal to an axis which it meets, this force being any function of the distance r of the element from the given axis. A special discussion is made of the case where the force varies inversely as r^2.

The general problem as here stated is suggested by Appell in his *Traité de Mécanique Rationelle,* Vol. I, Chap. VII. In connection with this problem he gives the three following equations which are directly obtainable from the results he has arrived at in his discussion of equilibrium of strings:

$$(1)\quad T\frac{dz}{ds}=c;\quad (2)\quad Tr^2\frac{d\theta}{ds}=k;\quad (3)\quad T=-\int F(r)dr-h=\phi(r),$$

in which T is the tension at any point, $F(r)$ the force, the z-axis the axis to which the force is normal, r and θ the polar co-ordinates in the xy plane, and c, k, h arbitrary constants.

Starting with these three equations and with the equation

$$(4)\qquad ds^2=dr^2+r^2d\theta^2+dz^2,$$

we have, on eliminating T,

$$(5)\qquad k\frac{dr}{rd\theta}=\pm\sqrt{\{r^2[\phi(r)]^2-c^2r^2-k^2\}},$$